HOMEWORK 10 - ANSWERS TO (MOST) PROBLEMS

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Section 4.5: Summary of curve sketching

4.5.11.

D : $\mathbb{R} - \{\pm 3\}$

I : No $x-$intercepts, $y-$intercept: $y = -\frac{1}{9}$

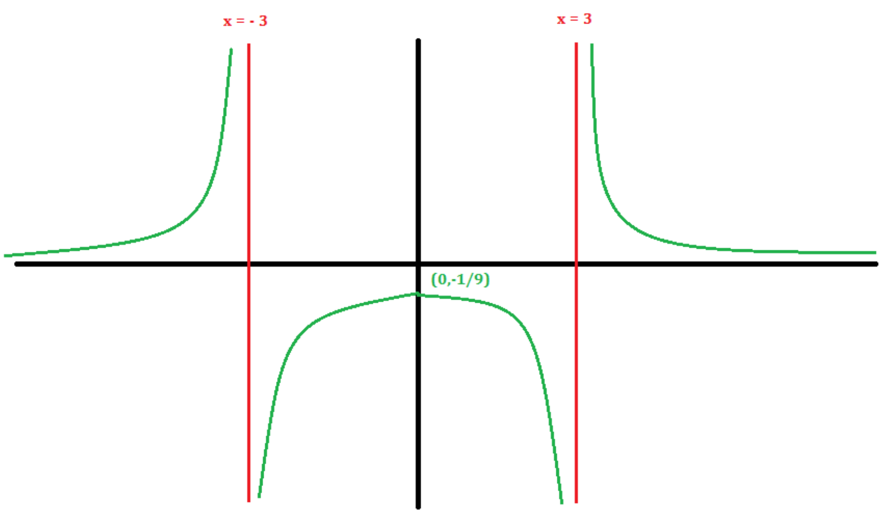
S : f is even

A : Horizontal Asymptote $y = 0$ (at $\pm\infty$), Vertical Asymptotes $x = \pm 3$

I : $f'(x) = -\frac{2x}{(x^2-9)^2}$; f is increasing on $(-\infty, -3) \cup (-3, 0)$ and decreasing on $(0, 3) \cup (3\infty)$. Local maximum of $\frac{-1}{9}$ at 0.

C : $f''(x) = 6\frac{x^2+3}{(x^2-9)^3}$; f is concave up on $(-\infty, -3) \cup (3, \infty)$ and concave down on $(-3, 3)$; No inflection points

1A/Homeworks/hw10graph1.png



4.5.31.

Note: First of all, f is periodic of period 2π, so we're only focusing on $[0, 2\pi]$.

D : $\mathbb{R}$

I : $x-$intercepts: $x = 0$, $x = 2\pi$ (basically you should get $\sin(x) = 3$, which is impossible), $y-$intercept: $y = 0$

S : Again, f is periodic of period 2π. Also, f is odd.

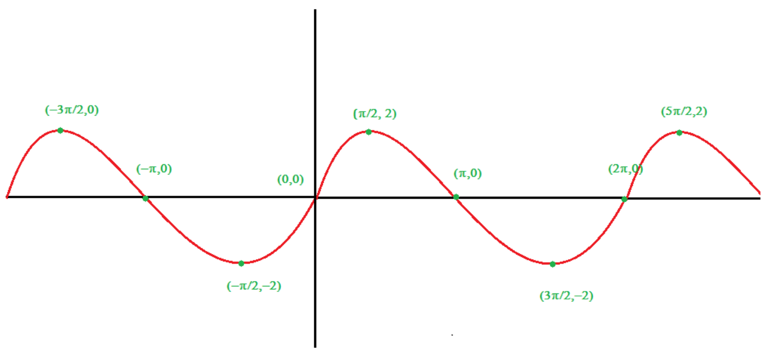
A : No asymptotes

Date: Monday, April 11th, 2011.

I : $f'(x) = 3\cos(x) - 3\cos(x)\sin(x) = 3\cos(x)(1-\sin^2(x)) = 3\cos^3(x)$; Increasing on $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$; Decreasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$. Local maximum of 2 at $x = \frac{\pi}{2}$. Local minimum of -1 at $x = \frac{3\pi}{2}$.

C : $f''(x) = -9\sin(x)\cos^2(x)$; Concave down on $(0, \pi)$ and Concave up on $(\pi, 2\pi)$. Inflection point $(\pi, 0)$

1A/Homeworks/hw10graph2.png



4.5.41.

D : $\mathbb{R}$

I : No $x-$intercepts, $y-$intercept: $y = \frac{1}{2}$

S : No symmetries

A : Horizontal Asymptotes: $y = 0$ (at $-\infty$), $y = 1$ (at ∞)

I : $f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$, so f is decreasing on $\mathbb{R}$

C : $f''(x) = \frac{e^x e^x - 1}{e^x + 1^3}$ (multiply numerator and denominator by $(e^x)^3$ after simplifying), so f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Inflection point at $(0, \frac{1}{2})$

1A/Homeworks/hw10graph3.png

4.5.47.

Note: : First of all, $\boxed{f \text{ is periodic of period } 2\pi}$, so from now on we may assume that $x \in [0, 2\pi]$

D : We want $\sin(x) > 0$, so the $\boxed{\text{domain is } (0, \pi)}$

I : $\boxed{\text{No } y\text{−intercepts}}$, x−intercepts: Want $\ln(\sin(x)) = 0$, so $\sin(x) = 1$, so $\boxed{x = \frac{\pi}{2}}$

S : Again, f is periodic of period 2π

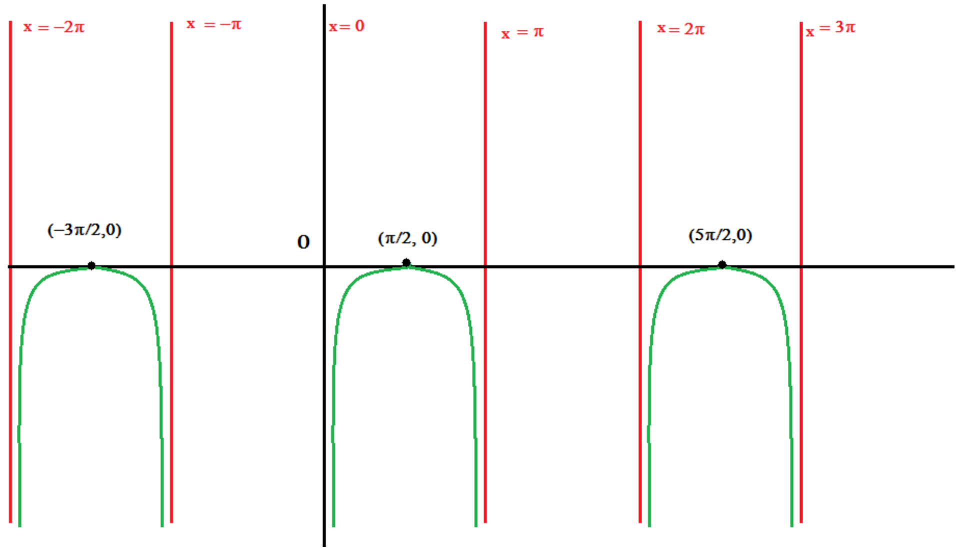
A : No horizontal/slant asymptotes, but $\lim_{x \to 0^+} \ln(\sin(x)) = \ln(0^+) = -\infty$, so $\boxed{x = 0}$ is a vertical asymptote. Also $\lim_{x \to \pi^-} \ln(\sin(x)) = -\infty$, so $\boxed{x = \pi}$ is also a vertical asymptote.

I : $f'(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$, then $f'(x) = 0 \Leftrightarrow x = \frac{\pi}{2}$, and using a sign table, we can see that $\boxed{f \text{ is increasing on } (0, \frac{\pi}{2}) \text{ and decreasing on } (\frac{\pi}{2}, \pi)}$.

Moreover, $\boxed{f(\frac{\pi}{2}) = \ln(1) = 0}$ is a local maximum of f.

C : $f''(x) = -\csc^2(x) < 0$, so $\boxed{f \text{ is concave down on } (0, \pi)}$.

1A/Homeworks/hw10graph.png



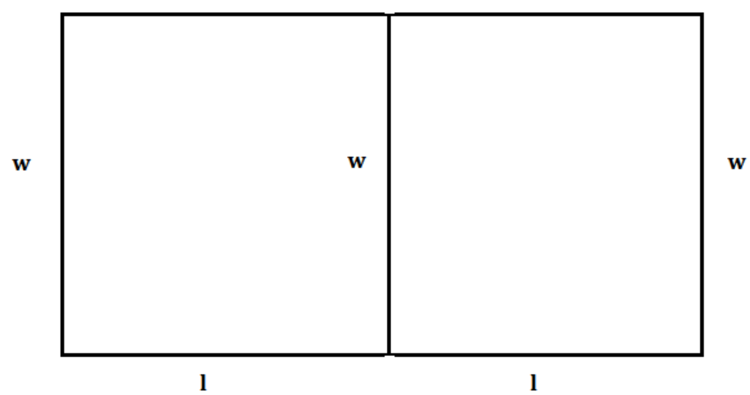
Section 4.7: Optimization Problems

4.7.3.

- Want to minimize $x+y$
- But $xy=100$, so $y=\frac{100}{x}$, so $x+y=x+\frac{100}{x}$
- Let $f(x)=x+\frac{100}{x}$
- $x>0$ (x is positive)
- $f'(x)=0 \Leftrightarrow 1-\frac{100}{x^2}=0 \Leftrightarrow x^2=100 \Leftrightarrow x=10$
- By FDTAEV, $x=100$ is the absolute minimum of f
- Answer: $\boxed{x=100,\ y=\frac{100}{100}=1}$

4.7.11. The picture is as follows:

1A/Practice Exams/Fence.png



- Want to minimize $3w+4l$
- But $2lw=1.5$, so $l=\frac{0.75}{w}$, so $3w+4l=3w+\frac{3}{w}$
- Let $f(w)=3w+\frac{3}{w}$
- $w>0$
- $f'(x)=0 \Leftrightarrow 3-\frac{3}{w^2}=0 \Leftrightarrow w^2=1 \Leftrightarrow w=1$
- By FDTAEV, $w=1$ is the absolute minimum of f
- Answer: $\boxed{w=1,\ 2l=1.5}$

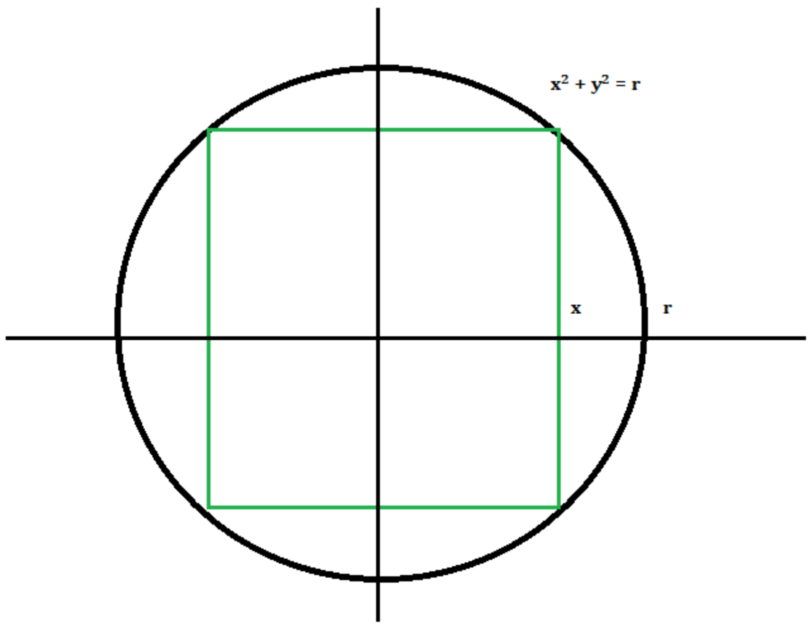
4.7.19.

- We have $D=\sqrt{(x-1)^2+y^2}$, so $D^2=(x-1)^2+y^2$
- But $y^2=4-4x^2$, so $D^2=(x-1)^2+4-4x^2$
- Let $f(x)=(x-1)^2+4-4x^2$
- No constraints
- $f'(x)=2(x-1)-8x=-6x-2=0 \Leftrightarrow x=-\frac{1}{3}$
- By the FDTAEV, $x=-\frac{1}{3}$ is the maximizer of f.
- Since $y^2=4-4x^2$, we get $y^2=4-\frac{4}{9}=\frac{32}{9}$, so $y=\pm\sqrt{\frac{32}{9}}=\pm\frac{4\sqrt{2}}{3}$

- Answer: $\boxed{\left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right) \text{ and } \left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right)}$

4.7.21. Picture:

1A/Homeworks/hw10opt1.png



- We have $A = xy$, but the trick here again is to maximize $A^2 = x^2y^2$ (thanks for Huiling Pan for this suggestion!)
- But $x^2 + y^2 = r^2$, so $y^2 = r^2 - x^2$, so $A^2 = x^2(r^2 - x^2) = x^2r^2 - x^4$
- Let $f(x) = x^2r^2 - x^4$
- Constraint $0 \leq x \leq r$ (look at the picture)
- $f'(x) = 2xr^2 - 4x^3 = 0 \Leftrightarrow x = 0$ or $x = \frac{r}{\sqrt{2}}$
- By the closed interval method, $x = \frac{r}{\sqrt{2}}$ is a maximizer of f (basically $f(0) = f(r) = 0$
- Answer: $\boxed{x = \frac{r}{\sqrt{2}},\ y = \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}}}$

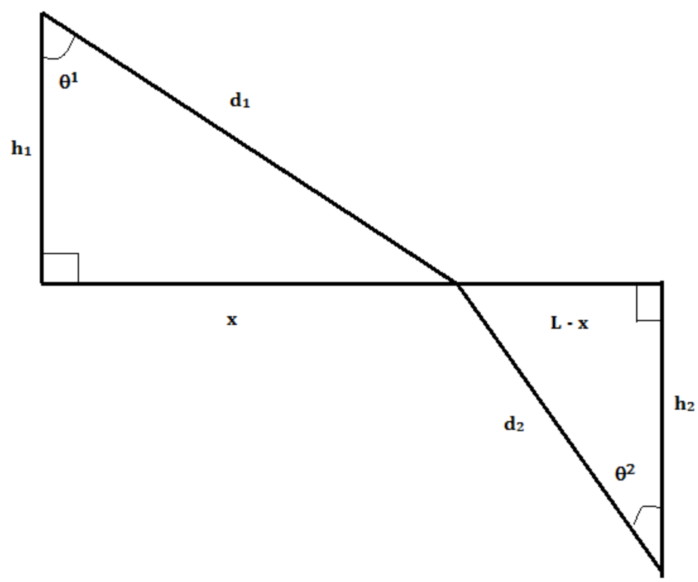
4.7.30.

- Let w be the width of the rectangle, and h the height of the rectangle.
- We have $A = wh + \pi(\frac{w}{2})^2 = wh + \frac{\pi}{4}w^2$, but $w + 2h + 2\pi\frac{w}{2} = 30$, so $2h + \pi w + w = 30$, so $h = \frac{30-(\pi+1)w}{2}$. Hence $A = w(\frac{30-(\pi+1)w}{2}) + \frac{\pi}{4}w^2$
- Let $f(w) = w(\frac{30-(\pi+1)w}{2}) + \frac{\pi}{4}w^2$
- Constraint: $w > 0$
- $f'(w) = 15 - \frac{(\pi+2)}{2}w = 0 \Leftrightarrow w = \frac{30}{\pi+2}$ (there's a big cancellation going on!)
- By FDTAEV, $w = \frac{30}{\pi+2}$ is the maximizer of f
- Answer: $\boxed{w = \frac{30}{\pi+2}, h = \frac{15}{\pi+2}}$

4.7.53. (*a*) $c'(x) = \frac{C'(x)x - C(x)}{x^2}$. When c is at its minimum, $c'(x) = 0$, so $C'(x)x - C(x) = 0$, so $C'(x) = \frac{C(x)}{x} = c(x)$, so $\boxed{C'(x) = c(x)}$, i.e. marginal cost equals the average cost!

4.7.63. (thank you Brianna Grado-White for the solution to this problem!)
The picture is as follows:

1A/Homeworks/hw10opt2.png



Here, h_1 and h_2 and L are fixed, but x varies.
Now the total time taken is $t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}$.

Now, by the Pythagorean theorem: $d_1 = \sqrt{x^2 + h_1^2}$ and $d_2 = \sqrt{(L-x)^2 + h_2^2}$, so we get:

$$t(x) = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(L-x)^2 + h_2^2}}{v_2}$$

And

$$t'(x) = \frac{x}{v_1\sqrt{x^2 + h_1^2}} + \frac{x - L}{v_2\sqrt{(L-x)^2 + h_2^2}} = \frac{x}{v_1 d_1} + \frac{x - L}{v_2 d_2}$$

Setting $t'(x) = 0$ and cross-multiplying, we get:

$$v_1 d_1 (L - x) = v_2 d_2 x$$

So, by definition of $\sin(\theta_1)$ and $\sin(\theta_2)$), we get:

$$\frac{v_1}{v_2} = \frac{d_2 x}{(L-x)d_1} = \frac{\frac{x}{d_1}}{\frac{L-x}{d_2}} = \frac{\sin(\theta_1)}{\sin(\theta_2)}$$

Section 4.9: Antiderivatives

4.9.7. $F(x) = 5\frac{x^{\frac{5}{4}}}{\frac{5}{4}} - 7\frac{x^{\frac{7}{4}}}{\frac{7}{4}} + C = 4x^{\frac{5}{4}} - 4x^{\frac{7}{4}} + C$

4.9.24. $f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + A$, so $f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Ax + B$

4.9.33. $f(x) = -2\sin(t) + \tan(t) + C$, but $4 = f(\frac{\pi}{3}) = -\sqrt{3} + \sqrt{3} + C = C$, so $f(x) = -2\sin(t) + \tan(t) + 4$

4.9.33. If $f''(\theta) = \sin(\theta) + \cos(\theta)$, then $f'(\theta) = -\cos(\theta) + \sin(\theta) + C$.
$f'(0) = 4$, so $-1 + 0 + C = 4$, so $C = 5$.
Hence $f'(\theta) = -\cos(\theta) + \sin(\theta) + 5$.
Hence $f(\theta) = -\sin(\theta) - \cos(\theta) + 5\theta + C'$.
$f(0) = 3$, so $-0 - 1 + 0 + C' = 3$, so $C' = 4$.
Hence $\boxed{f(\theta) = -\sin(\theta) - \cos(\theta) + 5\theta + 4}$

4.9.61. $a(t) = 10\sin(t) + 3\cos(t)$, so $v(t) = -10\cos(t) + 3\sin(t) + A$, so $s(t) = -10\sin(t) - 3\cos(t) + At + B$

Now, $s(0) = 0$, but $s(0) = -10(0) - 3(1) + A(0) + B$, so $-3 + B = 0$, so $B = 3$

So $s(t) = -10\sin(t) - 3\cos(t) + At + 3$

Moreover, $s(2\pi) = 12$, but $s(2\pi) = -10(0) - 3(1) + A(2\pi) + 3 = A(2\pi)$, so $A(2\pi) = 12$, so $A = \frac{12}{2\pi} = \frac{6}{\pi}$

So altogether, you get: $\boxed{s(t) = -10\sin(t) - 3\cos(t) + \frac{6}{\pi}t + 3}$

4.9.74. First of all, the acceleration of the car is $a(t) = -16$, so $v(t) = -16t + C$. We want to find $v(0) = C$, so once we find C, we're done!

Let t^* be the time when the car comes to a stop.
Then $v(t^*) = 0$, so $-16t^* + C = 0$, so $C = 16t^*$. So once we find t^*, we're done!

Now we know that $s(t^*) - s(0) = 200$, but $s(t) = -8t^2 + Ct + C'$, so $200 = -8(t^*)^2 + Ct^* + C' + 0 - C(0) - C' = -8(t^*)^2 + 16t^*t^* = 8(t^*)^2$, so $8(t^*)^2 = 200$, so $(t^*)^2 = 25$ so $t^* = 5$ (assuming time is positive)

Whence $\boxed{v(0) = C = 16t^* = 80}$